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THE MATHEMATICS TEACHER

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CIRCLES THROUGH NOTABLE POINTS OF THE TRIANGLE

BY PROFESSOR RICHARD MORRIS

Rutgers University, New Brunswick, N. J.

The Geometry of the Circle and the Triangle has been most fascinating since the inception of the science. Through the centuries some very important points and lines have been discovered and many of them have been named. And these points and lines have become the basis for the extension of the study, with the result that very beautiful and valuable theorems and properties have been developed. This paper will present a study in the application of many of these notable points in determining circles. The reader should construct his own figures for the most part. The material used is classic, but it is hoped that the applications will prove helpful to teachers of Geometry. We will not restrict ourselves to notable points entirely, but will use some other points determined from them by notable lines and other circles. The method of erecting perpendiculars and of bisecting angles and lines will be assumed.

POINTS

In the triangle whose vertices are A, B and C, we have the circumcenter O, the orthocenter H, the centroid G, the incenter I, the excenters I', I'' and I''', opposite A, B and C respectively, the mid-points, A', B' and C', of the sides opposite A, B and C, respectively, the feet of the altitudes AD, BE and CF. Other points will be introduced as needed.

CIRCLES ON THREE POINTS

The circumcircle of the triangle is one such circle. In the extension of the study of Geometry through the centuries, other

points were found to lie on this circle, but we shall think of it temporarily as a three-point circle.

The circle throught points B, O and C will have its center O_1 on the perpendicular bisector of BC. The center of the circle on C, O and A is O_2 on OB' produced, and the center of circle AOB is O_3 on OC' produced. These three centers determine a triangle, $O_1O_2O_3$, similar to the pedal triangle DEF. It is easily shown that the angle at O_1 equals $180° - 2A$, since angle BOC equals $2A$. If R denotes the radius of circle ABC, the radius of circle BOC equals $R/2 \sec A$, of COA, $R/2 \sec B$, and of AOB, $R/2 \sec C$.

The circle on BHC has its center at O_a on OA' produced. The center of CHA is at O_b on OB' and of AHB at O_c on OC'. The triangle $O_aO_bO_c$ is similar to triangle ABC, since the angle at O_a equals angle A, angle at O_b equals angle B. It can be shown that O_a is the reflection of O in line BC. (See Fig. 1.) Join

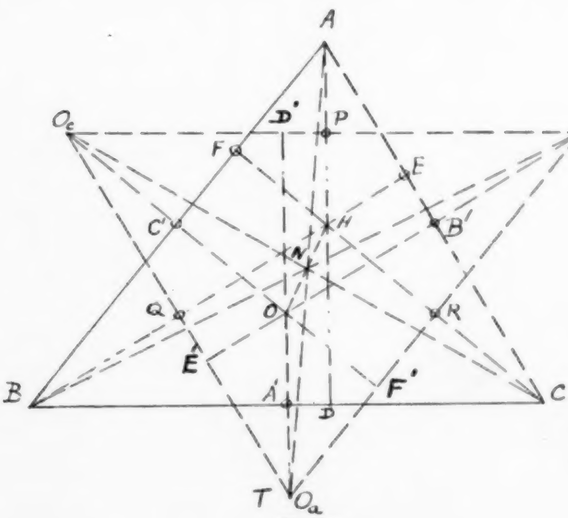

FIG. 1

A to the mid-point N of the segment OH, and produce this line to meet OA' produced, denoting the intersection by T. Triangles ANH and TNO are congruent, angle, side, angle, and hence OT equals AH, and thus A' is the mid-point of OT, since AH equals twice OA', the latter being a well-known fact. Also, TH equals

OA, being opposite sides of a parallelogram. But TB equals OB and TC equals OC. Hence T concides with O_a, both being the center of the circle BHC. Hence the circle BHC is the reflection of circle ABC in line BC. The radii of circles BHC, CHA and AHB equal R.

The circle on BGC has its center at O_{ag} on OA' produced, the center of CGA at O_{bg} on OB' and the center of AGB at O_{cg} on OC'. Let AA' remain fixed, and upon this line, with BB' and CC' as the other two sides, construct a triangle called the triangle of the medians. The triangle $O_{ag}O_{bg}O_{cg}$ is similar to this triangle of the medians, since the sides are respectively perpendicular.

The circle on BIC has its center at O_{ai} on OA' produced, the center of CIA is at O_{bi} on OB' and the center of AIB is at O_{ci} on OC'. The triangle $O_{ai}O_{bi}O_{ci}$ is similar to the triangle formed by joining the three excenters I', I'' and I''', since the lines joining the excenters are perpendicular respectively to the internal bisectors. The angle at O_{ai} evidently equals $\frac{B+C}{2}$, at O_{bi}, $\frac{C+A}{2}$, and at O_{ci}, $\frac{A+B}{2}$. The center O_{ai} is the midpoint of line II', since a line parallel to the base of a triangle, and passing through the mid-point of one side (the perpendicular bisector of BI), bisects the other side also. The point O_{ai} is evidently on the circumference of circle ABC, since the bisector of an angle passes through the mid-point of the intercepted arc and the line from the center perpendicular to a chord bisects the arc. The circle on BIC becomes a four-point circle, since it contains the point I'. Similarly, the circles CIA and AIB become four-point circles, containing I'' and I''' respectively. We can now speak of circle ABC as at least a six-point circle, since it contains three additional points, viz., O_{ai}, O_{bi} and O_{ci}.

The circles BOC, BHC, BGC and BIC are evidently coaxial, their centers being on OA' produced.

The circle on O_aOO_b has for its center C, that on O_bOO_c has A for its center and that on O_cOO_a has B for its center. This is evident by the property of reflection. The radius of each of these three circles equals R.

It can be shown that H is the center of the circle on $O_aO_bO_c$,

since $CO_a = HO_a = BO_a = R$, $CO_b = HO_b = AO_b = R$ and $AO_c = HO_c = BO_c = R$, by perpendicular bisectors and reflection.

Four-Point Circles

The points $A'OB'C$ are on the circle whose diameter is OC; similarly OA is the diameter of the circle on $AC'OB'$, and OB the diameter of circle on $A'BC'O$. These centers are the vertices of a triangle similar to ABC and equal in area to one fourth of it; it is congruent to the triangle $A'B'C'$.

Let P, Q and R be the mid-points of AH, BH and CH respectively. (See Fig. 1.) Then O_aH, O_bH and O_cH are the diameters of the circles on O_aQHR, O_bRHP and O_cPHQ respectively, since O_aO_b is perpendicular bisector of CH, O_bO_c of AH and O_cO_a of BH. The centers of these circles are the vertices of a triangle similar to both $O_aO_bO_c$ and ABC.

Segments AH, BH and CH are the diameters of circles on points $AEHF$, $BFHD$ and $CDHE$ respectively, with centers at P, Q and R respectively.

If in triangle $O_aO_bO_c$ the feet of the altitudes from O_a, O_b and O_c to the opposite sides are D', E' and F' respectively, then OO_a, OO_b and OO_c are the diameters of circles on the points $OE'O_aF'$, $OF'O_bD'$ and $OD'O_cE'$ respectively and their centers are the points A', B' and C' respectively. Evidently, the circles on AH and OO_a as diameters are equal, similarly for those on BH and OO_b, and also for those on CH and OO_c.

The circle on AB as diameter also contains the points D and E. Similarly, the circle on BC as diameter also contains points E and F, and that on AC the points D and F. But these three circles become eight-point circles, since four additional points are found on each circle. The perpendiculars from A upon the internal and external bisectors of the angle at B give two points on this circle, as also do the perpendiculars from B upon the bisectors of the angle at A. It can be shown that the points on the bisectors at B lie on the line $B'C'$. Let B_1 and B_2 be the points on the internal and external bisectors at B. The figure AB_2BB_1 is a rectangle, since the bisectors are perpendicular. The rectangle being inscriptible, angle $AB_2B_1 =$ angle ABB_1 and angle $AB_2B_1 =$ angle B_2B_1B, by parallel lines. Hence, angle $B_2B_1B =$ angle B_1BC and line B_2B_1 is parallel to BC through point C'. There are twelve such points, determined by

the perpendiculars upon the bisectors of the angles, such that four of them lie on the circle on AB as diameter, and four, not the same four, lie on line $B'C'$. Similarly, for the other eight points.

If perpendiculars are dropped from D, the foot of the altitude AD, to sides AB and AC, locating points D_1 and D_2 respectively, then AD is the diameter of the circle on points AD_1DD_2. Similarly, BE is the diameter of the circle on points BE_1EE_2, and CF the diameter of circle CF_1FF_2.

Similarly, by taking AG as diameter and dropping perpendiculars from G to the sides AB and AC, we get a four-point circle. Likewise, for the segments BG and CG. Similarly, lines AO, BO and CO may be used as diameters, as may AI, BI and CI. There may be found other points on these circles which lie on lines drawn from either end of the diameter and to which perpendiculars are dropped from the opposite end of the diameter. The number of such points is limited if we use only notable lines emanating from A.

The four feet upon AB and BC of the perpendiculars from D, E and F are concyclic. (See Fig. 2.) In triangle ABD,

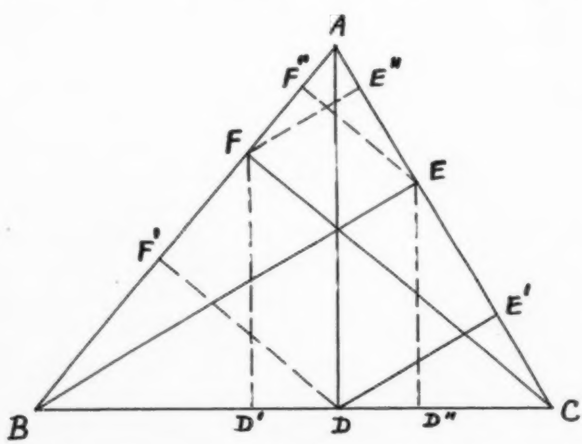

Fig. 2

$BF' = (BD)^2 \div AB$; in triangle BFC, $BD' = (BF)^2 \div BC$; in triangle ABE, $BF'' = (BE)^2 \div AB$; and in triangle BEC, $BD'' = (BE)^2 \div BC$. Hence

$$\frac{(BF')(BF'')}{(BD')(BD'')} = \frac{(BD)^2(BC)^2}{(BF)^2(AB)^2}.$$

But $(BD)(BC) = (BF)(BA)$ since $AFDC$ is a circle. Thus $(BF')(BF'') = (BD')(BD'')$ and points $F''F'D'D''$ are concyclic. In this proof, we use the theorem, and its converse, relating to a secant and tangent to a circle. Similarly, the four feet upon BC and AC are concyclic, as are the four feet upon AB and AC. If these three circles are distinct, their common chords, viz., the sides AB, BC and CA, should be concurrent. They are not concurrent, hence, these circles coincide, and the circle becomes a six-point circle. When Fig. 4 is explained, it will be possible to find six more four-point circles.

Definition.—If points B_1 and C_1 are taken on AB and AC respectively, such that angle AC_1B_1 equals angle ABC, line B_1C_1 is said to be antiparallel to line BC, with respect to angle A.

The points B_1C_1CB are concyclic. But the circle becomes a three-point circle if B_1 coincides with B, or C_1 coincides with C, or both B_1 and C_1 coincide with A. Again, if B_1 and C_1 coincide with F and E respectively, the circle becomes an eight-point circle. It can be easily shown that the line FE is antiparallel to side BC. If B_1 coincides with C', we get a four-point circle through three notable points.

Five-Point Circles

If AA' is used as a diameter, point D and the feet of the altitudes from A' upon AB and AC give three additional points upon this circle. Similarly, BB' and CC' are the diameters of five-point circles. The same type of circle is obtained in triangle $O_aO_bO_c$ if O_aP, O_bQ and O_cR are used as diameters.

The General Theorem

In triangle ABC (see Fig. 3) let A_1 be any point on BC. Take E, a point on AB, such that angle A_1EB equals a given angle α. On AA_1 as a chord, describe a circle in which shall be inscribed the angle α. Produce BC and AC, locating Y and F on the circle. The point E lies on this circle since angle F or Y is supplementary to angle AEA_1. To construct a five-point circle, given angle α and point A_1 on BC, lay off angle $BEA_1 =$ angle $A_1FC =$ angle $CYA = \alpha$. If $\alpha = 90°$, line AY coincides with

AD and lines A_1E and A_1F become orthogonal to AB and AC respectively. But these circles become four-point circles if A_1 coincides with B, or C, or D. If AA' is used as the diameter, we have the case first discussed under Five-Point Circles.

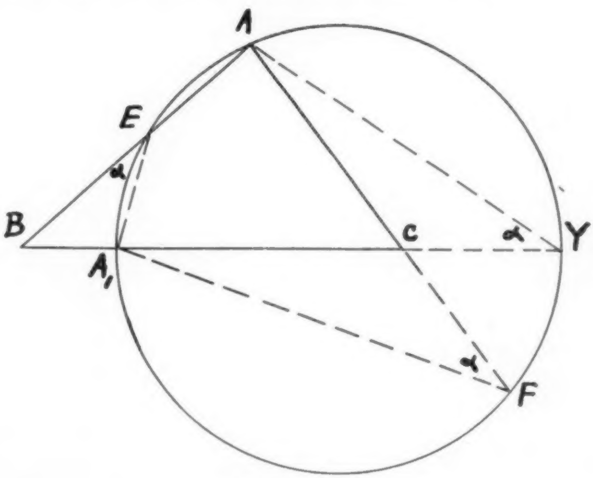

FIG. 3

The bisectors of angle A meet the side BC in two points which are the extremities of the diameter of a circle passing through A. Such a circle is called the Apollonian circle of the triangle. There are two other such circles for the triangle. We may at first think of them as three-point circles, but later when we learn that these three circles have two points in common, we see that they are really five-point circles.

The segments OH, GH, IH, $A'H$, $B'H$ and $C'H$ are the diameters of five-point circles, the additional points being obtained as the projections of these points upon the altitudes of the triangle.

SIX-POINT CIRCLES

If from A' and B' perpendiculars are drawn to the lines BC, $A'C'$, $B'C'$ and AC, we get a six-point circle on $A'B'$ as diameter. Similarly, $B'C'$ and $A'C'$ are the diameters of other six-point circles.

Definition.—If AX and AZ are two lines making equal angles with the angle bisector AI, these lines are called Isogonal Con-

jugates with respect to AI. It is known that points H and O lie at the same time on Isogonal Conjugate lines from A, B and C. Points related in this way are called Isogonal Conjugate Points of the triangle.

It can be easily shown that the three reflections, viz., H_a, H_b and H_c, of H in the sides of the triangle lie on the circumcircle of ABC, since the angle formed at a reflected point is supplementary to the opposite angle of the triangle. This makes the circumcircle a nine-point circle.

It can be shown that the mid-points of the segments formed by joining the excenters lie on the circumcircle of ABC. Each segment is the diameter of a circle passing through two vertices of the triangle, since right angles are formed at the vertices. The segment between the vertices is a common chord of the two circles, hence its perpendicular bisector locates the mid-point of the segment of the excenters. But this perpendicular bisector by its intersection with the two angle bisectors at the third vertex determines the diameter of a circle passing through the third vertex. And this diameter becomes the diameter of the circumcircle of ABC and hence the mid-point of the segment on the excenters lies on the circumcircle. Thus the circumcircle becomes a twelve-point circle.

General Theorem

See Fig. 4. Take points V_1, V_2, V_3, W_3, W_2 and W_1 on AB, BC and CA respectively, such that angle $HV_1B =$ angle $HV_2C =$ angle $HV_3A =$ angle $OW_1C =$ angle $OW_2B =$ angle $OW_3A =$ a given angle α. Triangles AV_1H and AW_1O are similar. Hence $AH : AO = AV_1 : AW_1$. Also triangles AOW_3 and AHV_3 are similar, and $AO : AH = AW_3 : AV_3$. Multiplying, we get $AV_1 \cdot AW_3 = AW_1 \cdot AV_3$, so that these points V_1, V_3, W_1 and W_3 are concyclic. Similarly, $V_1W_3V_2W_2$ are concyclic, as are $V_2W_2V_3W_1$. But these circles coincide, since the sides of the triangle are not concurrent, and we thus get a six-point circle. The center of this circle is at the intersection of the perpendicular bisectors of segments V_1W_3, V_2W_2 and V_3W_1. If $\alpha = 90°$, the points V and W become the feet of the altitudes of the triangle and the mid-points of the sides. The center of the circle, when $\alpha = 90°$, is N, the mid-point of segment OH. But in this latter case, the points P, Q and R also lie on this circle and we

have what is known as the nine-point circle. Referring to Fig. 1, the proof is as follows: Join A' to N and let this line cut AH at some point, say P'. Then triangles $A'NO$ and $P'NH$ are con-

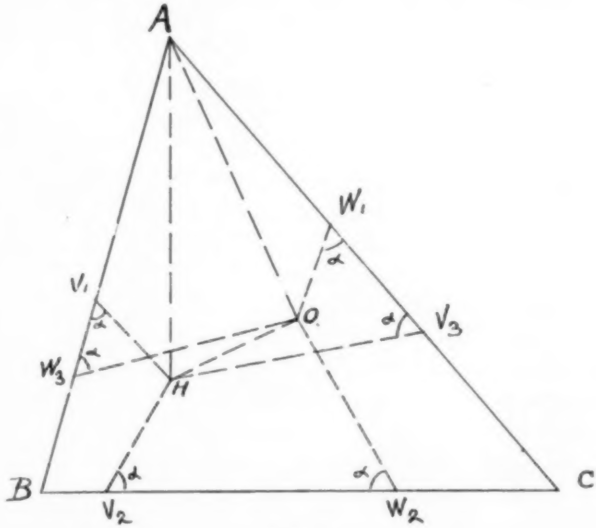

FIG. 4

gruent, angle, side, angle. Hence $P'H = A'O$ and P' coincides with P, the mid-point of AH. Thus $A'P$ is a diameter of the nine-point circle. Similarly, $B'Q$ and $C'R$ are also diameters of the nine-point circle. Evidently, the feet of the altitudes in triangle $O_aO_bO_c$ also lie on this circle, so that we get a twelve-point circle.

There are many other circles of interest that are related to the triangle and its notable points. But those which we have exhibited are sufficient to illustrate the abundance of the material along this particular line. A number of the theorems have been changed to suit the purposes of this article. In the preparation of this paper, the author has used *Modern Geometry* by C. V. Durell, *Modern Geometry* by Godfrey and Siddons, *College Geometry* by Altshiller-Court, *The Circle and the Sphere* by J. L. Coolidge.

THE WHY AND THE HOW IN ALGEBRA[1]

By HARRY B. MARSH

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One of the most important questions at the basis of mathematical understanding and discovery is contained in the single word, *Why*. The question *Why* indicates a most desirable attitude toward activity in any field of mathematics. It is especially desirable as a class-room attitude. When pupils are in such a state of mind that they wish to know the reasons for various processes and steps, they are at least on the road to real appreciation and enjoyment in mathematics. Unfortunately, so far as this situation is concerned, ours is a large country, and, speaking figuratively, too small a percentage of our pupils are from Missouri by nature. This, however, does not relieve us as teachers from doing all in our power to stimulate in the minds of our boys and girls the spirit of curiosity and inquiry. We should create and constantly encourage the questioning and even the challenging attitude that must ever know the reason why. We must realize that to be alive and alert, a class must be interested and active and that we are always more interested in activities in which we have a share. Hence it is important not only to secure a questioning attitude but also to translate that attitude into actual class discussion and participation. Only under such circumstances can the best and most constructive progress be made.

Even when, in some cases, it is not possible for pupils to formulate properly questions about new processes, it is essential that the teacher should still assume that the questioning attitude is there and that she should see to it that the reasons for the different steps are understood as they are presented.

In the development of new processes and new ideas, the *Why* of it all must be made clear. Needless to say, the presentation of a new process should be brought about, so far as possible,

[1] Presented at the Annual Meeting of the Association of Teachers of Mathematics in New England, December 3, 1927.

through the mutual activity of both teacher and pupils. In fact, most new processes and steps should be worked out together. Very few of them should be handed to the class. Whenever possible and advisable, pupils should be given the opportunity to suggest probable next steps. They should have the chance to enjoy the experience of recreating and having a part in new developments. This procedure not only increases interest but it tends to fix new proofs and processes more definitely in the mind.

When an explanation has been completed, the discussion should not be left without giving opportunity for further questions. Make it a point to encourage such questions. Keep in mind that no question is a foolish one if it is asked seriously and honestly. Pupils should feel free to ask questions about relatively simple things. The little things are likely to be the stumbling blocks, as they are likely to be overlooked. When we enter a room we sometimes stumble over a book or a ball. We rarely stumble over the piano. We meet it squarely, if at all.

If there is no response when you ask for questions, this may mean one of three things:

(*a*) The class understands the explanation thoroughly (but do not be too sure of this).

(*b*) There may not be that friendly and free feeling between teacher and pupils that encourages questions (under such circumstances I am sorry for both teacher and pupils, and especially for the latter).

(*c*) The class may wish to ask questions but not know just how to interpret its own difficulties and formulate them into questions.

In any one of these cases, the teacher should not assume that there are no questions to be asked. If there is any doubt in her mind as to whether the class has comprehended the meaning of her previous explanations, she should ask a few more questions herself.

Without the *Why* in algebra, the subject would become mechanical, dry, and uninteresting. Pupils would probably be able to follow directions and arrive at satisfactory results but they would lack a proper and necessary conception of the meaning of it all. The *Why* is absolutely essential if the subject is

to possess vitality and power and appeal. The *Why* is certainly necessary if we are to awaken in our pupils any latent ability for logical thought and reasoning that will help them to meet new situations and new problems successfully as they arise in later study and in later life. In all branches of mathematics, as well as in algebra, the question *Why*, with all that it implies, must be kept constantly in the foreground.

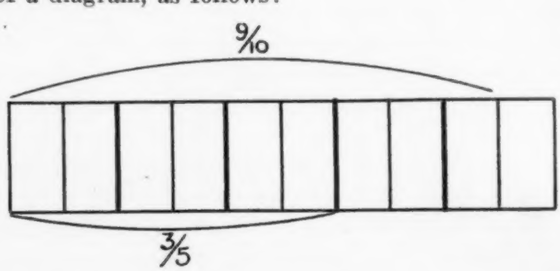
There is another question, however, that deserves its full share of consideration and it is contained in the single word *How*, for the various processes in algebra are to be used as well as understood. Algebra is a tool as well as a source of mental training and logical reasoning. Considered solely as a means of development in logical thinking and reasoning, algebra certainly has a place. But on that score alone it would not deserve all the time, attention, and energy that is given to it. After presenting the different steps of any particular process, the next question is how to carry out the process in the most effective and economical way, without, of course, losing sight of the reasons for the different steps.

In some cases these reasons should be brought up frequently to be sure that mere terms are not being used in a meaningless way. In other cases the *Why* would hardly need to be referred to again. Let me illustrate with two or three examples.

Let us suppose that the division of fractions is to be presented for the first time to a class in arithmetic and we have the example:

$$\frac{9}{10} \div \frac{3}{5}.$$

I should probably first present the division graphically by means of a diagram, as follows:



The quotient is $1\frac{1}{2}$ or $\frac{3}{2}$.

I might also change to a common denominator and show that

$$\frac{9}{10} \div \frac{3}{5} = \frac{9}{10} \div \frac{6}{10} = \frac{9}{6} = \frac{3}{2},$$

for just as 9 feet divided by 6 feet gives 9/6 or 3/2 as the quotient, so 9 tenths divided by 6 tenths will also give 9/6 or 3/2 as the quotient.

In any event I should make use of enough examples to bring out the principle that we can divide fractions by dividing the numerators for the new numerator and the denominators for the new denominator. This would have a familiar sound, for in the multiplication of fractions we learned to multiply the numerators together to get the new numerator and the denominators to get the new denominator. There is a possibility of confusion however in sometimes multiplying numerators and at other times dividing them. It is a good thing with younger children to connect up with former habits. So, in order to tie up to all their previous practice with and experience in the multiplication of fractions I should then make it clear that when dividing fractions we can invert the divisor and proceed as in multiplication.

Having once reasoned out this method, I should not refer to it again unless some special situation demanded it. I think that you will agree with me that when the *Why* of this arithmetic process has once been understood, the important consideration thereafter is *How*. The division of fractions then becomes a problem of manipulation. The *How* does and should occupy not only our chief attention but practically all of our attention.

In presenting the subtraction of signed numbers for the first time, I should first consider subtraction as the process of finding what number must be added to the subtrahend to produce the minuend as the sum. I should study with the class many examples like the following, leaving both the examples and the results on the board for inspection.

$$\begin{array}{ccccc} +12 & +9 & -15 & -6 & -2 \\ +7 & -10 & -8 & -14 & +9 \\ \hline \end{array}$$

I should prepare or find many others of the same kind and assign them to be done by the same method, that is, finding what number must be added to the subtrahend to produce the minu-

end. Working with the class, I should then gradually bring out the fact that all these examples can be made examples in addition by changing the sign of the subtrahend. Here again I believe that there is a distinct advantage in this change to a previous process, addition, for the habits and skills that were acquired in the development of the addition of signed numbers are then carried over bodily into the subtraction of signed numbers.

Thereafter I should carry on the subtraction of signed numbers on the basis of changing the sign of the subtrahend and proceeding as in addition. The *How* now comes into prominence. The subtraction of signed numbers is a matter of such frequent occurrence that the doing is the important thing. However, the *Why* must not be lost sight of entirely. One should not forget to stop pupils occasionally and have them show that they are still finding numbers which added to the subtrahend will produce the minuend.

And so with the reduction of fractions to lowest terms, the process must be clearly understood as the division of both numerator and denominator by the same number. Here the *Why* is very important. In a comparatively short time the underlying principle will be understood and accepted. When it is so understood, the question of whether or not the process is called cancellation or given some other name is not of much consequence. It is of great importance, however, that any time the pupil who uses the expression "cancelling the $(x-y)$'s" should understand and be able to explain that this really means dividing both numerator and denominator by $(x-y)$. Here the *Why* is very necessary and the question must be raised often. But pupils are to reduce fractions frequently. Hence the *How* must again receive its full share of attention.

Just one more illustration. In presenting the subject, fractional equations, for the first time, I should approach it something like this:

$$(1) \qquad \frac{7x+5}{6} - \frac{5(x-1)}{4} = \frac{1}{3} + \frac{x+7}{2}.$$

This equation would be much easier to solve if I could change in some way from the fractional form to an equivalent equation without fractions. And I should not hesitate to say, if I could

"clear the equation of fractions." Working with the class I would soon find that if I multiply each fraction by 12, I would get rid of all denominators. We would then proceed to multiply by 12 as follows:

$$(2) \qquad \frac{12(7x+5)}{6} - \frac{12\cdot 5(x-1)}{4} = \frac{12}{3} + \frac{12(x+7)}{2},$$

and then get:

$$(3) \qquad 2(7x+5) - 15(x-1) = 4 + 6(x+7).$$

Then I would go through many other examples where the denominators are all numerical, picking out with the class in each case the number by which we must multiply each fraction in order to get rid of all the denominators. I would then introduce examples in which there are literal denominators, requiring in every case, for a while at least, that each fraction and term be multiplied by the lowest common denominator. Finally, however, I should allow pupils to pass directly from the given fractional equation (1) above, to equation (3), without explaining all the multiplications.

When the process is completed, the equation is certainly cleared of fractions. If you object to that terminology, call it what you will. I know of no better way of speaking of the process of getting rid of fractions than "clearing of fractions." I will go as far as anyone in insisting that the terminology must be understood and in challenging at any time any pupil who uses this expression to explain just what he means.

The *Why* here is gaining in relative importance over the *How*. The process, however, is easily seen. Pupils can soon visualize the operation which provides for each denominator dividing in turn into the lowest common denominator and leaving the numerator to be multiplied by the quotient. Call the process what you will or leave it nameless if you wish. I see no objection to calling it "clearing of fractions" so long as the corresponding process has been carefully developed and understood.

With the foregoing two or three illustrations I have attempted to bring out the point that both the *Why* and the *How* in algebra are important and that their relative importance varies with different operations and processes. Irrespective of its final position and standing, the *Why* must always receive first consideration, for it throws the light of understanding on the *How* that is

to follow. However, as algebra is to be used as well as understood, the *How* must necessarily occupy more of the student's time and attention than the *Why*, for the simple reason that he is spending most of his time doing. But we must not become so attached to the elementary steps that go into the development of the *Why* of any process that we shall insist on their entire appearance and repetition every time the process is used, when these very steps themselves lead to reasonable, shorter methods.

This brings me directly to the consideration of the simplification of equations and to the use of the process popularly or unpopularly known as transposition. This process has been receiving no little attention and, in fact, considerable opposition of late. Transposition, however—placing or carrying across—is used elsewhere in algebra as well as in equations and with little or no protest. In dealing with positive and negative exponents we show that a factor can be carried across the fraction line, that is, it can be transposed, if the sign of its exponent is changed. We see that this vertical transposition also involves both carrying across and a change of sign. Having shown that $a^{-n} = 1/a^n$, we do not hesitate to apply the principle directly to examples involving negative exponents. We could of course insist that in examples like $\frac{2x^{-2}y^{-3}}{3x^{-5}z}$ and others of the same type, both numerator and denominator should be multiplied by such factors as would remove the negative exponents. But is this refinement worth while and do you wish to carry it consistently through all later work? Common practice says No. Vertical transposition is the generally accepted procedure.

While there may be some objection to vertical transposition, the real protest develops when transposition becomes horizontal instead of vertical. As a result, there are many teachers to-day who fear that if they use the process known as transposition—meaning of course horizontal transposition—they may be committing an unpardonable mathematical sin and there are others also who, if they do use the process, feel that it must be suitably camouflaged to escape detection.

I wish to come out squarely and unhesitatingly in favor of the process, transposition, whether it runs north and south or east and west. The process is too valuable and time-saving to lose. The fact that it has stood the test of years is not necessarily a

strong argument for its retention. Many good things have stood this test, only to be replaced by others. The fact, however, that the process is both economical and reasonable is worth serious consideration.

Equations should of course be simplified at first and for some little time by the process of adding the same quantities to each side or subtracting the same quantities. After this method of simplification has been carried on for some time, such examples themselves are likely to bring out the fact that the resulting equations show quantities appearing on the opposite side from which they did at first and with the reverse sign. Left to themselves pupils always have discovered, and probably always will discover this. The *Why* of the process is perfectly understandable.

Given the equation:

$$8x - 6 = 3x + 9,$$

if we subtract $3x$ from each side of this equation we shall have

$$8x - 3x - 6 = 9,$$

the $+ 3x$ becoming $- 3x$ on the left side by the process of subtraction and the $+ 3x$ disappearing from the right side.

We can make the application more general by showing that if $a + b = c$, the $a = c - b$, and that if $a - b = c$, then $a = c + b$.

We have many theorems in geometry which, once proved, are thereafter directly applied to subsequent work. After the different steps of the proof have been built up to a logical conclusion we do not expect pupils to keep repeating these steps every time they wish to use the principle involved. We have demonstrated the conclusion for the purpose of using it. I should be willing to make the principle of transposition a theorem in algebra and, once proved, refer to it as I would to a theorem in geometry and use it in the same manner.

To be accepted transposition must satisfy two tests: (1) Is it mathematically sound and (2) Does it have an advantage over other methods?

In answer to (1), the principle of transposition can be established by logical mathematical proof.

In answer to (2), does it have an advantage over other meth-

ods?, let me in turn ask you here two or three questions: If you have an equation like this to solve,

$$8x-7+3x=5x+17,$$

are you going to fuss around adding quantities to both sides or subtracting from both sides, or are you going to get the result quickly by inspection,

$$6x=24, \quad x=4.$$

And when you are solving quadratic equations and wish to get the x's on one side of the equation and the constant term on the other, how are you going about it? Or, how are you going to get all the terms on one side of the equation so that you can solve the quadratic equation by the formula? What method do you prefer to use? What method will you give your pupils?

Shall we expect our pupils then to putter along through high school and college using a cumbersome method that we ourselves would not use after it has served its purpose?

Transposition is so reasonable, so understandable, so simple, so practical, and so inevitable, that pupils should be given the advantage of it as soon as they see the *Why* of it. Its application to the quicker and more practical solution of equations should be made as soon as possible in the light of both mathematical expediency and common sense. Those who wish to use roundabout processes in place of more direct and just as sound ones should do so if they get a corresponding increase in satisfaction; but, in my opinion, they should not require or expect hundreds of pupils to take the unnecessary detours with them.

On page 66 of Bulletin 1921, No. 32, of the Department of the Interior, "A Summary of the Report by the National Committee on Mathematical Requirements," we find that the Committee recommends the abandonment, so far as possible, of certain terms outright, such as "aggregation for grouping," "vinculum for bar," etc. So far as "clear of fractions" and "cancel" and "transpose" are concerned this phrase is used "at least until the significance of the terms is entirely clear." I assumed that their objection to these terms and processes is removed when this condition of clearness is fulfilled. However, to get light on this point I wrote to Professor J. W. Young of Dartmouth College, chairman of the National Committee. I quote from my letter:

"I should like to ask whether the committee had in mind the elimination of these processes as well as the terms."

"Was it the sense of the committee, for instance, that pupils should always simplify equations by the long process of adding quantities to both sides of an equation or subtracting quantities from both sides?"

"Am I right in interpreting the phrase 'at least until the significance of the terms is entirely clear' to mean that after the simplification of equations by the long process has been made entirely clear as well as its shortening by the principle of transposition, there would then be no objection to the use of transposition?"

I am giving herewith Professor Young's reply which I am using with his permission:

"Replying to your letter of November 18, I am very sure that the National Committee did not at all intend to eliminate the processes to which you refer. The processes in question are, of course, essential for an easy technique in Algebra and it would seem to be very foolish to attempt to eliminate them.

"The Committee's recommendations applied merely to the terms and I doubt if a majority of the Committee really favored the elimination of the terms. The Committee was, as I recall the discussion, unanimous in feeling that the *premature* introduction of such terms as "cancel" and "transpose" does a great deal of harm, in that if introduced too early the pupil uses them and the corresponding processes mechanically, without understanding, and it is then inevitable that he will use them erroneously. My own feeling is that even such terms as you mention can well be retained provided they are not introduced until the significance is entirely clear. As you say, pupils will discover for themselves the processes involved and when they once thoroughly understand them it is certainly an advantage to have suitable terms to describe these processes briefly. Your interpretation as exhibited in your letter seems to me entirely correct."

The *Why* and the *How* in algebra are both important because development and practice in algebra are both important. Neither must be neglected. Each must maintain its proper relation to the other. We must keep constantly in mind that mathematics is a logical science but we must not insist on mathematical nice-

ties to the extent that we are passing up opportunities for effective practices that are both sound and efficient.

Noticeable changes have been made in recent years in mathematical content. The National Committee and the College Board Commission have made distinct contributions in revising the mathematics curriculum and in bringing it up to date. Weeks and months were spent by these groups in preparing their documents and reports. Much traditional material that had been in our courses for years was removed and new material and new applications were added. The reports have been welcomed throughout the country by mathematics teachers everywhere. We must remember, however, that all this was the work of mathematicians for mathematicians. It was almost entirely among ourselves.

In the coming years other changes must and should be made. Changing aims and changing ideals will undoubtedly change to a marked degree present courses and present practices in secondary education. Tradition will not have such a strong hold on educational policies as it has had in the past. Broad and sweeping educational statements and theories that were accepted without question yesterday are being carefully investigated and checked up to-day. This is an age of testing, of scientific testing. Topics and subjects must justify themselves in both content and method in terms of actual accomplishment. In common with other parts of the curriculum, mathematics courses and procedure will be more and more studied, questioned, and challenged, and by persons from without the subject. How can we best meet this testing and this challenging? How can we best justify the claims we make for our subject and for its rightful and adequate place in the curriculum? It seems to me that we shall weaken our position decidedly if we are found concerning ourselves so seriously and so subject-mindedly on petty details, little refinements, and unnecessary niceties. We must meet the tests and criticism that are sure to come, not from the standpoint of the purist but in the broader spirit of reasonableness and usefulness. We must recognize the fact that to be most effective any science must be founded and built upon two fundamental principles: first, clear and thorough development, and then the most reasonable and effective practice.

EXTRANEOUS DETAILS

By VERA SANFORD

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It is universally admitted that the mathematical problems which arise in business, economics and science are seldom neat composites of information that is both necessary and sufficient for the obtaining of a solution. Many "real life" problems contain information that is not needed and they often lack necessary details. The two situations are not mutually exclusive and each challenges the wit and ingenuity of the person who tries to solve the problem in a way peculiar to itself.

It is evident that textbook problems in algebra or in arithmetic approximate real problems more closely when the parts necessary to a solution must be sifted out of a mass of apparently relevant data or when they must be supplemented by information obtained elsewhere. The inclusion of extraneous information and the omission of needed data have characterized verbal problems from the very beginning of their history, but to-day writers are perhaps more aware of the value of the idea than ever before, and accordingly a discussion of some of the possibilities of this device may not be ill-timed.

The value of the two types of problems lies in the fact that they compel a student to exercise judgment in differentiating between details which are important and those which are not, or else they oblige him to search for the additional information that he needs. In a way, then, the one type represents the same sort of thinking as does the other, but on a higher level. In the one case, the student selects the necessary idea from a collection of facts appearing before him. In the other, he selects the needed relationship from those with which his brain is stored. At times, these relations are selected automatically. For example, consider the following problem: How many pounds are there in a quarter of a ton? This problem belongs in the "missing data" class but the pupil is scarcely conscious of this fact when he uses the "2,000 pounds = 1 ton" which he once mem-

orized. On the other hand, suppose a boy travelling in England steps on a weighing machine and receives a ticket marked 10 *stone* and he wonders how many pounds that is. It may be that this information is in his mental stock; it probably is not. The use of an unfamiliar unit makes him at once aware that the problem lacks an important detail.

Extraneous data may be present in a number of forms. Among these may be listed descriptive details of name, place, time, etc.; numerical details quite irrelevant to the problem; numerical details that seem to be essential; and quantitative relations stated redundantly. At the same time, it would be well to consider problems in which essential mathematical relations are obscured by the phraseology of the problem, as in the time-honored question, "How old is Ann?"

The use of details of time, place, and name to make a problem vivid is probably as old as are problems themselves. The division of booty among a sea captain and his sailors could belong only in the buccaneering days of Sir Francis Drake. The discussion of a bond given "18th day of the 5th month" places the arithmetic in which it appears in Philadelphia.[1]

> "Charles learned ten chapters in the Bible, and Lydia learned eighteen; how many more did Lydia learn than Charles?"

This could come from no place so well as New England and the title page shows that the work was printed in Portland, Maine.[2]

Sometimes the details of a problem are unusually significant. For example, Nicholas Pike, writing from Newburyport, Massachusetts, gave a case in which A and B set out from Providence and Newburyport respectively travelling toward each other. The problem asks when and where they met and the answer which accompanies it not only gives the number of miles each travels but says that their place of meeting is "near Ames's at Dedham." This seems unimportant in itself, but a contemporary almanac shows that Ames's was the tavern.

At times, writers borrow names of famous personages. The "Greek Anthology"[3] contains a dialogue between Pythagoras

[1] Stephen Pike, *The Teachers' Assistant*, Philadelphia, 1834.

[2] Oliver Welch, *Improved American Arithmetic.* (First edition, 1834; 1841 ed. Portland.)

[3] A collection of puzzles and epigrams compiled about 500 A.D.

and Polycrates, while a work of more recent date concerns an important writer of the period.

> "Suppose Mr. Peter Parley wants a loan at bank of \$994.50 for 30 days at which time he expects to be able to refund it from the profits of his story book, and that Mr. Paywell endorses it: what sum must be specified in the note to obtain that loan?"[4]

Beside this obvious type of interest-getting detail, there is the more subtle use of subject matter of current importance. Ten years ago Liberty Bonds were much featured in Junior High School work. The first flight of the Los Angeles over New York resulted in an epidemic of problems concerning dirigibles. It would not be surprising if Lindbergh's trip to Paris has already furnished the material for a textbook problem. The study of books of other days shows that this capitalizing of current questions is not new, but it also shows that the resulting problems are not lasting ones.[5] The Holland Tunnel may thrill a New York boy to-day, but by the time books now in press have attained their next revision, a vehicular tunnel under the Hudson will be as much a commonplace as is Brooklyn Bridge or the Atlantic cable. These problems have vital use but only transitory value, and, in a short time, they lose their specific details and deal with general cases, or else they disappear entirely.

Another type of extraneous verbal detail has the purpose of making a fictitious problem appear to be reasonable. A single illustration of two forms of the same problem will illustrate this point although there are few types of problems that have not received similar treatment at one time or another. The simple form of the problem is from the work of the German writer, Adam Riese (1522); the other is from that of Humphrey Baker, an Englishman, writing in 1568.

> "Someone says, 'God greet you, all you thirty companions.' One of them answers, 'If there were as many of us again and half as many more, then there would be thirty of us.' The question is how many were there?"

> "A man hauing his eye sight somewhat altered, began to tell and reckon a certayne number of birdes to be in all 18.

[4] Roswell C. Smith, *Arithmetic on the Productive System* (First edition, 1841; second edition, Rochester, N.Y., 1846), p. 196.

[5] See *The History and Significance of Certain Standard Problems in Algebra*, Vera Sanford, New York, 1927.

> His Companion that had a clearer sight, beholding wel the birds, answered him, that there were not 18, but saide he, if ther were twice so many more as there are, there should be as many more abou 18, as ther be now less than 18. The question is to know how many birdes there were in all."

When this use of verbal detail is analyzed it is apparent that its purpose is the double one of arousing the student's interest and of convincing him that the subject that he studies has contacts with real situations. A study made by L. L. Hydle and summarized by Clapp [6] shows that of two problems of identical content, mathematically the one containing information that is the more difficult to visualize is the more difficult. Ligda reports similar findings in his *Teaching of Algebra.*

It has been the writer's experience that these verbal details have a greater appeal to immature pupils than to older ones. Yet any generalization of this sort must be discounted by the popularity of Raymond Weeks's ***Boys' Own Arithmetic.*** Consider the following cases, for example:

> "On the Manakin Road lives a Dog who barks for 40 minutes whenever someone goes by at night. What is the smallest number of passers-by that will keep him barking all night? Allow $10\frac{1}{2}$ hours."

> "Mrs. Superbia MacManus, age unknown, having little to do, joined the Browning Club of MacIvorsville. On her way to attend the second fortnightly meeting for December, she slipped and fell in front of the well known Tonsorial Parlors of Lonnie Frazell, and had to return home for repairs. She was 43 minutes late in arriving at the meeting. During every $1\frac{1}{3}$ minutes of the time she was late, one-sixteenth of her character disappeared. How much remained?"

No grown person can resist them, but some literal-minded fifteen-year-olds would think them too trivial for consideration. Their disgust may be traced to an idea expressed by a high school senior in these terms: "I am not interested in who A and B were, or what they were doing, or why B quit work. It's the thinking that counts. It seems to me that, for high school seniors, it makes little difference whether the problems are practical or not. It doesn't change our thinking."

[6] "Some Recent Investigations in Arithmetic," *The First Yearbook of the National Council of Teachers of Mathematics.*

At this point, it may be well to discuss problems in which mathematical relationships are obscured in the wording of the question. An extreme example of this type reads as follows:

> "Two automobiles twenty miles apart are approaching each other, each travelling ten miles an hour. A bee which flies at the rate of fifteen miles an hour starts at the radiator of one automobile and flies back and forth between their radiators until the autos meet. How far does the bee fly?"

A better illustration occurs in Roantree and Taylor's *Arithmetic for Teachers.*

> "The Joneses drove over to visit the Browns. They found on reaching there that the Browns had been gone 11 minutes to visit the Joneses, so started back home. The Browns found that the Joneses had been gone 15 minutes, so set out to return. The two parties met midway between their homes at 4 o'clock. At what time did the Browns leave home?"

In each of these problems, essential details are expressed in terms of bewildering descriptions.

In the case of genuine problems from business, economics, and science, verbal descriptions are often absolutely essential to enable the student to work out a proof. The long discussion often required to properly orient the pupil with respect to the question constitutes one of the most serious objections to the use of genuine problems in class work.

Among the dangers incident to the use of descriptive detail is the fact that things familiar to a child in one locality are foreign to his cousin in another. A city child was reading a problem that asked for the number of ordinary cows worth so and so much that could be bought for the $3,000 paid for the Duchess of Queensberry. Unfamiliar with the names of registered cattle, she exclaimed, "Why, was the Duchess a cow?" Many problems which illustrate topics of social or economic interest are valuable for the very fact that they take a child's thoughts into situations which are new to him. On the other hand, it is unfortunate to localize a textbook question in a place interesting only to a few. There is another danger in that the details of a problem may distract a child's attention. In a well-known Arithmetic, a page is devoted to a girl's efforts to earn a college

education. Her method was that of buying materials wholesale and baking bread for a steadily growing group of customers. On concluding this page of exercises, one class asked how Jane could do all this and still do her school home work. In another case, in which a boy bought a camp equipment from his uncle who allowed him a discount of 10 percent, a pupil shifted an odd half cent of discount to the boy's favor on the plea that if it were a disinterested storekeeper the boy would lose, but she was sure his uncle would give him the benefit of the doubt.

On the whole, then, the use of descriptive detail is a valuable tool which loses its effectiveness when used too constantly.

The second type of extraneous detail is that of irrelevant numerical data. It is clear that only the weakest intellects would be misled by a problem concerning the sale of six weeks'-old puppies at so many dollars each. In fact, in a recent attempt to persuade a class to decide on the unimportant details of a problem before starting to solve it, no one realized that the age of the puppies was extraneous. It seemed to them equivalent to "a certain sort of thing" and they refused even to dignify it with the name "non-essential." The only excuse for this sort of information is to lend reality to a problem and it is subject to the same dangers as is the descriptive detail.

The third type of irrelevant data is that of numerical details that appear to be relevant to the problem. For example, a certain teacher delights to ask his students how many gallons of water run off a barn roof during a storm in which the depth of rainfall is one inch. He states the length of the barn, its height, and its width, the length of the ridgepole, the length of the rafters, the pitch of the two sides of the roof, and the height of the gable end above the ground. The rain comes straight down. In many cases, the solution discloses rain that is perpendicular to both sides of the roof at once.

This type of extraneous numerical data belongs on a higher level than does the type mentioned previously. It presents genuine difficulties to the student and it probably represents the most effective use of unnecessary detail.

The fourth type is the inclusion of a restatement of an essential relationship. It must be confessed that this logical possibility does not occur to appreciable extent, if at all.

Of the various groups of extraneous details, the most valuable

is that in which the given apparently useful information is actually of no account. The descriptive details have value only when they challenge the student's interest by their semblance of reality or by their reference to situations which he knows. A wealthy child who saw no point in the pipes filling a cistern, worked with interest when the teacher suggested that the method applied equally well to a swimming pool similar to the one at her house. Taken by and large, these details can be made vital parts of the questions, but they must not be overdone.

Textbook problems that lack essential details are even more interesting but they are fewer in number. The missing information may be things that the pupil is supposed to know, as the number of feet in a yard. It may be material that he cannot be supposed to know, but which he must find from data given elsewhere in his text or in an almanac or encyclopædia. Again it may be details of depreciation, costs or what not that cannot be ascertained, and which leave the student with the statement, "If I know thus and so, I could answer the question by doing this and that." Finally, the details that are omitted may be the relations between variable quantities and the problems may be truly indeterminate.

It is quite evident that if the student is expected to supply missing details, those details should be readily accessible to him through tables given in his text or through other material that is at hand.

On the other hand, although the printing of the information elsewhere in the text makes the solving of the problem a somewhat routine matter, it is often a waste of time to send a pupil to an atlas or to the *Statistical Abstract* to find the length of the Amazon or the size of the cotton crop.

Numerical details that cannot be ascertained by the study of reference material leave the student in the position of being unable to find a numerical result. The situation is unsatisfactory to him but again it approximates a real situation. Problems of this type are more frequent than one would at first suppose. They are often phrased so convincingly that both the student and teacher are certain that all needed details are present.

An example of this type asks whether it is better to build a garage for $400 or to rent one for $4 a month. The methodical students in a certain seventh grade class reported that it was

wiser to build. The more brilliant ones didn't know. They questioned whether the man would have to buy the site for the garage, whether the increased value of his property would compensate for his increased insurance, what his new taxes would be, what the depreciation would be, and so forth. Incidental to this discussion, New York building laws were cited, insurance rates quoted, a debate was organized, the class inquired whether a garage *could* be built for that price, and at length decided that it would be wiser to rent rather than to build. Yet it would be safe to guess that the authors of the problem expected the answer to be given the other way.

The third type of data that may be lacking is that in which the problem is indeterminate. These were of frequent appearance in the sixteenth century, but they went into disuse in elementary textbooks when their solution was reduced to complicated algebraic rules. It is to be hoped that representatives of this group of problems will again appear in our algebras. Consider, for example, this question from the work of Humphrey Baker (1568):

> "A man hath geuen vnto 20 workefolk 20 *s*. that is to say, vnto men, women, and boyes: vnto men he gaue 20 pence a peece, vnto women 15 pence, and vnto boyes he gaue 8 pence. The question is to knowe how many men: how many women: & how many boyes there were in all."

We would give the equations as follows:

$$x + y + z = 20,$$
$$20x + 15y + 8z = 240.$$

Since a third equation is lacking, an infinite number of sets of values will satisfy the two relationships that appear here. The fact that x, y, and z must be integers, however, in order that the problem may have meaning, restricts these answers to the single set 2, 8, and 10. Other examples could be cited in which several groups of answers are permissible.

The use of irrelevant data and the omission of essential details is less common in geometry than it is in algebra or arithmetic. Yet there is an interesting tendency in recent texts to utilize extraneous information as in the following case:

> "Two chords AB and CD intersect at M, arc $AC = 50°$, arc $CB = 30°$, arc $BD = 100°$. How large is angle AMC?"

The omission of necessary facts is prominent in the questions whose answer is "You cannot tell."

The use of these problems in geometry can be defended on the ground that, like similar cases in arithmetic and algebra, they make for discriminating, reflective thinking.

The inclusion of data that are unnecessary and the omission of facts that are needed, then, is a powerful and valuable tool, but one that must be used with discretion. Non-essential data are not always extraneous from a mathematical point of view. Such details may hamper a child's ability to visualize the question or they may increase his interest in the problem but they should be viewed in the light of preliminary hurdles that have little effect on the work ahead. On the other hand, the omission or the inclusion of details that are mathematical in nature make the student more nearly a discoverer in his own right.

GEOMETRY HUMANIZED [1]

A School Play in One Act

By ERMA SCOTT

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Characters

Geometry.
Plane and Solid.
Triangles P D Q and I O U.
Parallelogram B R A G.
Polygon A D U N C E.
Triangles A B C and A′ B′ C′.
Circle O.
The Teacher.
The Ghost of Euclid.
Mary, Elizabeth, Robert, Lewis, Lloyd, Charles, Florence, and several other high school pupils.

Introduction

This little play was written for a sophomore class to give for an assembly program. At the time there was no intention of publishing it, but the play held the interest of the high school audience so well that the writer decided to put it on the market, thinking that other geometry teachers might find it useful for the same purpose.

The costumes were made in the school shops by members of the class. The triangles, the parallelogram, and the other polygon were light-weight wood frames covered with building paper. These the boys and girls held in front of them. Only the feet were visible, except when one'e turn came to speak. Then he peered around from behind the figure. The original intention was to make openings for the faces in all the figures.

Geometry carried a large paper-covered rectangle on which was drawn a partially opened book. For the costume of Plane,

all sorts of plane figures were cut from colored paper and fastened to a dress. A circle of stiffer material with an opening large enough for the face was tied to the head. Solid had an appropriate suit made from heavy boxes such as one can get at any grocery store. One large box slipped over the head and rested on the shoulders. Another cubical box fitted over the opening in the large one and formed the head. This, too, had an opening for the face. Both boxes were covered with bright-colored paper.

Circle O was made up of two large circles cut from wallboard. These were nailed to cross pieces of wood, allowing enough space between them for a small boy to roll the circles from the inside.

Prologue

Geometry appears in front of the curtain accompanied by his two children, Plane and Solid.

Geometry (*with dignity*): I am Geometry, and these are my children, Plane and Solid. (Plane and Solid curtsy.) I can see by the expressions on your faces that some of you have never heard of me. With a few of your number I may claim a passing acquaintance, but I regret that this acquaintance has not developed into friendship and understanding.

For the first time I have left my home and assumed the form of a human. At least I thought I looked human until I came here, for I tried to mould myself into a shape resembling one of my admirers—an extremely bookish geometry teacher.

I come from the land of Truth, a country only partially explored. Many have spent their lives trying to find it. Some do not recognize it when they see it, while others think they possess the whole land, when they are standing on a small corner facing the opposite direction. There are also those who are too indolent to seek it for themselves. They are content to accept the opinions of the more ambitious. I myself am familiar with only a minute area of my own country, but in that limited district I am sure of the ground.

I may boast of the wisdom of the ages, for I am thousands of years old. Civilizations have perished while I have remained the same. My friends have been the renowned of all generations—Thales, Pythagoras, Euclid, and others too numerous to mention.

All over the world you will find monuments to my importance.

In fact I have had a hand in every great building that has ever been constructed. It really ought to be beneath my dignity to boast of friends and achievements, but for the sake of my children I am willing to humble myself.

They claim that never in all their lives have they received such treatment as they have at your tongues. You have been accused of murdering the King's English, but that is a minor offense compared with what you have done to my family. If all you have said about us were true, our reputations for reliability would be ruined. In view of this alarming situation, we have decided to hold a conference of the outstanding figures of plane geometry to discuss measures for restoring our good names and warning those who have slandered us.

Scene I

The scene is in a mathematics room of a high school. At the left is the teacher's desk. At the right the chairs have been pushed back. About the room are grouped all the figures except Circle O, who is late in arriving.

The time is evening when everyone except the janitor has left the building. The figures are engaged in conversation when the curtain rises.

Triangle I O U: Polygon A D U N C E, you don't look quite natural. Weren't you originally composed of three triangles? Now I see you have four. You had better try some reducing exercises.

Polygon A D U N C E: Yes, I have added a triangle since you saw me in Smith's Essentials. You notice any gain in me more than you do in some people, because the addition of a triangle changes my shape.

Parallelogram B R A G: (Proudly) I guess that is true. When I grow larger it isn't so noticeable. By increasing my base and my altitude ever so slightly I can become many square inches larger. If, for instance, my base is 8 inches and my altitude 3 inches, my area will be 24 square inches. But if my base is increased to 10 inches and my altitude to 4, my area will be 40 square inches. I shall be nearly twice as large, but the change in my shape will not be apparent.

Triangle P D Q: (Aside) Pretty well satisfied with himself, is

he not? (Aloud) In my opinion, Circle O can put it all over the rest of us. Did you ever see such a well-proportioned figure? If you increase his diameter, you increase his circumference in the same ratio, for his circumference is always approximately three and one-seventh times his diameter.

Polygon A D U N C E: Yes, Circle O is a smooth figure. One can't accuse him of having any rough edges. But who would want to be so smooth and characterless. I prefer to have some individuality. All circles look alike to me. Now if I cared to, I could easily be reduced to a triangle, or I could be made to look very nearly like a circle. Even close observers would exclaim, "Why, there is no difference at all between them." But there is a limit to everything and the circle is mine. I can't quite bring myself to the place where I am willing to lose my own personality.

Parallelogram B R A G: (With sarcasm) A fine speech, brother Polygon. I suppose we all have our good points, and everybody except you and me has some bad ones.

Polygon A D U N C E: Well, I think it is better to have a few bad points than not to have any points at all. I have always thought of a circle as rather an unstable figure, easily swayed from one side to another.

Triangle A B C: By the way, isn't it about time for Circle O to be showing up? One never can tell when he is going off on a tangent.

Parallelogram B R A G: He is a great protracter. I'll wager both my diagonals that he will come rolling in an hour late.

Triangle A' B' C': Looking at it from all my angles, we haven't wasted any time. It does me good to talk everybody over, in a friendly spirit of course. I get so full I have to hold my sides to keep my altitudes erect.

Triangle P D Q: Sometimes I think people get away with a lot of meanness under the cover of friendship. (A noise is heard outside.) Hark! I believe Circle O is coming now.

Parallelogram B R A G: (Speaks as Circle O rolls in and stops near right entrance.) We had about given you up, Circle O. We hope you had no trouble on the way.

Circle O: I did have some tire trouble. About five miles from here I intersected a secant and punctured my inner tube, so I had to come in on the rim.

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Polygon A D U N C E: That is too bad. I've never picked up a secant. I'm surprised that you did with chorded tires.

Parallelogram B R A G: We had better get down to business. As you all know, we have met to discuss geometry students in general, and the ones in this school in particular. The situation doesn't appear alarming to me, but if it is serious, as most of you seem to think, I am willing to do my share in bringing about a reform.

Polygon A D U N C E: (Emphatically) I assure you I wouldn't have taken the trouble to come here if I hadn't considered it necessary to preserve the honor of Geometry.

Circle O: Do you think it as bad as that?

Triangle A' B' C': Polygon knows what he is talking about. As I was coming through the halls, I peeked into a room where there were a lot of figures on the board. I shouldn't have recognized them if they hadn't been labeled. There were circles with centers nowhere, and with circumferences everywhere. Right triangles with no right angles, isosceles triangles, all lopsided, and parallelograms with no two sides parallel.

Triangle I O U: But that isn't the worst. Skepticism is creeping into the school. This morning I heard an argument between a boy and his teacher. The boy insisted that there was a possibility that parallel lines if produced far enough would meet. In vain the teacher tried to convince him that if the lines did meet they would not be parallel. It is enough to make Euclid rise from the dead.

Parallelogram B R A G: (Stubbornly) I still insist there is no great cause for alarm. I can't say I have been shown any disrespect. (Emphasis on "I.") Boasting is not in my line, but I really believe the boys and girls like me. They are always glad when they get to parallelograms. I have heard them say, "I like parallelograms better than any other figures in the book."

Polygon A D U N C E: I don't want to be uncomplimentary, but this is no time for covering up things. I can tell you why they all like you. It is because you are easy. They all like easy things. "Nice and easy" is a favorite expression these days. It doesn't take them long to see through you. (Boastfully) I have a great many more possibilities, if I do have to say it myself.

Circle O: My opinion is concurrent with that of Parallelogram B R A G. I don't care to be considered so deep. I'd rather be an all around good fellow—easy to meet, and entertaining. I frankly admit that I want people to like me, and I court the favor of all who come within my radius.

Triangle A′ B′ C′: It seems to me that we are avoiding the point. To make a long story short, the pupils of this school have lied about us, drawn cartoons of us, and in other ways have threatened to destroy our dignity and reputation for truthfulness.

Circle O: Those are grave charges. I think we ought to visit one of these geometry classes, so each one of us can see for himself what has been going on.

Triangle P D Q: That is the most sensible suggestion that has been made yet. I move that we do that very thing.

Circle O: We shall have to hurry before the janitor locks us in.

Scene II

This scene is in the classroom the morning after the conference of the figures. It is nearly time for the first period class. All the pupils except a few stragglers are in their places when the curtain rises. Some are talking, others are trying to get their lessons at the last minute. The teacher is at her desk, making out the attendance slip.

Mary and Elizabeth enter from the right and take seats in the front row.

Mary: I can't understand these propositions about similar triangles at all.

Elizabeth: Neither can I. I never could understand proportion.

Teacher: Wait a few minutes, then I will explain it to all of you at the same time.

Robert: (Enters lazily dragging his feet, and takes a seat next to the aisle. Speaks as he enters.) When is spring vacation going to be?

Teacher: I don't know; they haven't informed us yet.

Robert: It can't come too soon to suit me. I'm getting all tired out working so hard. (Sprawls in his seat. The class is amused.)

Lewis: (Enters breathlessly, glances at his watch, is tripped by Robert, stumbles and drops books.) Cut it out!

Teacher: Lewis, can't you be more quiet?

Lewis: Yes, sir.

Teacher: (Addressing class) How many of you had trouble with today's lesson? (Nearly all raise hands.)

Lewis: I had a lot of trouble, but I worked on it until midnight, and I'm sure I can prove it.

Teacher: When did you begin to study, Lewis?

Lewis: At 11:45. (Laughter from class.)

Teacher: Come! Come! That will do!

Lloyd: I know it too. May I prove it first?

Mary: May I sharpen my pencil?

Teacher: No, you should have done that before you came to class.

Mary: How could I when I just broke it?

Teacher: Let us have quiet. I hear someone coming. I think we are going to have visitors. (Triangles A B C and A′ B′ C′ enter from right and take a position at the back between the desk and the chairs.)

Teacher: Good morning.

Triangle A B C: We beg your pardon for intruding. We are here in the interests of Geometry, Plane Geometry in particular, and we would like to see what kind of work your pupils are doing.

Teacher: Very well, we will go on with the recitation. We are now studying similar triangles.

Triangle A′ B′ C′: Ah, that is of special interest to us, as I happen to be Triangle A′ B′ C′, and this is my close associate, Triangle A B C. If you don't mind, I think it would be more satisfactory to let us take charge of the recitation.

Teacher: That will be all right.

Triangle A′ B′ C′: (Turning to class) Cecil, do you recognize us?

Cecil: (Frightened) Yes, sir, you are Triangles A B C and A′ B′ C′.

Triangle A B C: What do you know about us?

Cecil: (Doubtfully) You can be proved congruent.

Triangle A′ B′ C′: Take another look at us. Do you still insist that we can be proved congruent?

Cecil: Yes, sir, you have the three angles of the one equal respectively to the three angles of the other.

Triangle A B C: What does congruent mean?

Cecil: It means equal in all respects.

Triangle A′ B′ C′: Do we look equal in all respects?

Cecil: No, you have the same shape but not the same size.

Triangle A B C: (With disgust) Can anyone tell us how we are related?

Lewis: You are similar.

Triangle A B C: That is much better. Can anyone prove us similar when our sides are respectively proportional? (Several hands are raised.)

Triangle A B C: Robert, let us hear from you.

Robert: (Walks up to figure confidently.) If two polygons have their sides respectively proportioned, they are similar triangles. Given the triangles A B C and A′ B′ C′, (hesitates) with A B = A′ B′, A C = A′ C′, and B C = B′ C′.

To prove the polygon congruent.

Proof: Place the polygon A′ B′ C′ upon triangle A B C so that A B will equal A′ B′. (One straight line and only one can be drawn through two given points.) Then B C will equal B′ C′. (One straight line and one only can be drawn parallel to a given line.)

Triangle A B C: Take your seat. Not a thing you've said about us is true.

Triangle A′ B′ C′: We must not be too hasty in drawing conclusions. We'll call upon Lewis. (Lewis walks up proudly and makes a perfect recitation, then takes his seat feeling very much pleased with himself.)

Triangle A′ B′ C′: That is what I call good work.

Charles: I'd like to know if he can prove it with the letters changed?

Lewis: Sure I can. Do you want to hear me?

Triangle A B C: I'll call in Triangles I O U and P D Q; I told them to wait outside. (Triangle A B C leaves through door at the left, and Triangles I O U and P D Q enter.) (Lewis starts the proof again, calling the letters A B C and A′B′C′ as before.)

Triangle A′ B′ C′: (Interrupts impatiently) Take your seat. That isn't right at all.

Lewis: I'd like to know why. I got it from the book.

Triangle A B C: Just as I feared. Lewis has been memorizing.

Lewis: (Grumbles as he goes back to his seat) What's the use in buying a book if it ain't any good?

Triangle I O U: Ah, I see my friend Parallelogram is coming. (Class shows signs of pleasure.)

Lloyd: We're sure glad to se you. We all hate similar triangles.

Triangle P D Q: Such impertinence! That is because you do not understand us. We are the most misunderstood figures in Plane Geometry. The boys and girls of this age have no sense of proportion. They will go to any extremes to prove all proportions mean. With their antecedents it was different. If they didn't master their lessons, they knew only too well what the consequents would be.

Parallelogram: Boys and girls, what do you know about me?

Mary: Your opposite sides are parallel.

Elizabeth: Your opposite sides are equal.

Charles: Your opposite angles are equal.

Robert: Your diagonals bisect each other.

Parallelogram B R A G: I don't see anything wrong with this class. You Triangles scare the pupils, and I don't wonder at it. I've been connected with you all my life. We are composed of the same elements—straight lines and angles. Yet I confess you are sometimes too much for me.

Polygon A D U N C E: (Enters from right) What is all this fuss about: Children, you surely know me. I am Polygon A D U N C E. Florence, can you tell me what is the sum of my exterior angles?

Florence: What page are you on?

Polygon A D U N C E: What page am I on?

Florence: Yes, what page in Smith's Essentials?

Polygon A D U N C E: I must be obtuse, but I can't see what difference that makes. If you must know, I am on page 64.

Florence: (Brightening) Then the sum of your exterior angles is two straight angles. I can always prove the propositions better if I know what page they are on.

Polygon A D U N C E: (In exasperation) Well, I'll be bisected! This gives me a concave feeling. Pardon my rough speech. I have seen and heard about all I can stand. (Notices Circle O, who has come in from the right.) Circle O, do you have anything to say?

Circle O: No, I feel out of my locus. I've been standing behind the door trying to compass what has been going on, but I can't get a straight edge on the meaning of it all. If I stay around here much longer, I'll lose faith in the axioms.

(At this point the ghost of Euclid appears at the left. He is carrying a large roll of paper.)

Ghost: I am the ghost of Euclid. I have been listening and thinking. Like most conferences, this one seems to have accomplished nothing definite. You have been so occupied with thoughts about yourselves, your petty prides and jealousies, that you have forgotten that your coming together was for the good of all.

Still, the meeting has not been without results. You must have observed that Geometry can be interesting and thought-provoking at the same time. Where it has failed to be, there has been a reason. (Points an accusing finger at the teacher.) Geometry, the teacher, and the pupil have been considered incommensurable quantities. They have had no common unit of measure. Now it can be shown that the difference between commensurable and incommensurable may be less than one billionth or a trillionth of any positive value.

When you understand that Geometry can be humanized and that both teacher and pupil even now are human, then you can see that they are all quantities of the same kind and hence you can get a ratio between any two of them. Expressed in fractional form, their ratios would be as follows: (Unrolls paper and reads)

$$\frac{\textit{Geometry}}{\textit{Pupil}} = \frac{\textit{Geometry}}{\textit{Teacher}} = \frac{\textit{Pupil}}{\textit{Teacher}}.$$

It can be proved that these three ratios are equal. If Geometry to the teacher is uninteresting, then Geometry to the pupil will be uninteresting, and the pupil to the teacher will be uninteresting. Also since two equal ratios form a proportion and since in any proportion the product of the means is equal to the product of the extremes, if the value of the pupil is zero, then the teacher can amount to nothing. Likewise if the value of the teacher is zero, then Geometry will amount to nothing. Therefore when either the pupil or the teacher amounts to zero, nothing can be done with Geometry. A word to the wise is sufficient. I would like to have a private conference with the teacher.

ABILITY GROUPING OF STUDENTS IN SENIOR HIGH SCHOOL MATHEMATICS

By M. Bird Weimar

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Ten years ago the department of mathematics of Wichita High School began grouping students according to ability into three groups—honor, medium, and slow. At first we used an intelligence test as a part of the basis of classification. We have used the Stone Reasoning Test, the Otis Intelligence Test, and the Terman Test. With or without an intelligence test we have always used the judgment of the teacher who had the pupils the previous semester and the one receiving the pupils. Not always have we given an intelligence test but we do feel that we lose some of the certainty of placing and at times find our first judgment wrong without the test. A student may be very good in the mechanics of elementary algebra, doing the original reasoning required there in a satisfactory manner, and yet lack the power of analysis required in geometry, advanced algebra, and trigonometry. Students naturally group themselves according to their analytic and reasoning power and an intelligence test discovers this grouping more readily than the regular classroom procedure.

However, we have a flexible program so that students who fail to measure up to their group or are superior to it may be transferred to another at any time during the first period of six weeks, without other hardship to them than a possible change of teacher. As a department, we all feel that no student of medium ability who is intellectually lazy should be allowed to be in a plodding, slow-thinking group where he injures both the group and himself; for he can think out the problems and theorems more quickly, he can do the written work more rapidly, he can understand the principles more readily. If he remains in a slow group, he then continues to develop habits of laziness while, to the unobservant, he appears to be a capable, industrious, alert student who gets from the course all that he should.

We group all students in the tenth grade and in eleven B (first half of the eleventh year) very carefully according to ability. Beyond that we cannot group by any hard and fast lines because we have fewer groups and the students have more choice of subjects, some of which are not repeated; but as far as possible, we have the ability grouping even in the advanced classes.

Two of our teachers feel that the slow student loses much by not being allowed to cope with the bright student in mathematics. I was interested to note the testimony of students as given in Miss Parkhurst's book on Education according to the Dalton Plan, who say that the bright students are irritated to have to wait for repeated explanations for the benefit of their slow classmates in the ordinary class while the slow students especially appreciate the individual plan because then they can have the help and time they need without annoying the stronger students. Personally, I feel that their being given a method of presentation of work suited to their capacity and being allowed to match their talents with those of equal ability far outweighs any inspiration received from mingling with the bright students where too often they do not receive any inspiration but rather discouragement. A slow student in a class with others of his own ability certainly does not waste his time as he does in a class where the teacher and the bright students discuss topics of value to the latter. The brilliant student gets not only inspiration but a better understanding of the work; for there is an insight gained through the difficult problem and the application of mathematics not possible through the simpler work; yet this work is too often beyond the comprehension of the dull student.

We use the same textbooks for all groups and, as far as possible, the same problems. The fundamentals of the course are required of all and as much more as the class is capable of doing. In geometry the teachers of the department have compiled a syllabus and exercise book that we use in connection with the state text, that all students may have plenty of originals suited to their ability. We also have lists of typed problems in algebra.

The opportunity group is composed of the slow students, students slow in comprehending the work, slow in original thinking, slow in working out the problems and geometric exercises.

Among them we often find those who do not retain well what they have finally grasped or appeared to grasp. This group taxes the utmost tact, discernment, endurance, patience of the teacher every day of the semester to keep the students active, interested, to make the work worth while and keep the students eager to use all their ability to get the most possible from the course. A teacher should be at her best every hour of the school day but to be at her best with the slow, plodding group takes more of her life blood than it does in the medium and superior groups.

To this slow group we give much group help, much individual help. We explain all new processes and principles very thoroughly and when occasion seems to demand it, more than once. We encourage questioning. When a student has forgotten an underlying principle of algebra or a theorem of geometry, we have it explained anew and pause to have him state the truth again. This helps much in gaining individual attention when new work is presented and requires less reteaching. If the new work is of average difficulty, we have the class write out the new work under our supervision while they have drill problems of similar nature and review work to do independently. We make special efforts to teach our slow students how to study and to develop in them a spirit of mastery. Plenty of drill on fundamentals of the subject with a little of its theory, its history, its applications seems the best means of making mathematics worth while to the slow students. We should never forget that from among such a group have come some of the geniuses of the human race. I have just read in the life of Pasteur that when the time element entered into his tests he could not rank well but with the time element omitted he could do well. This is true not only of such a scientific genius as he, but of many of the great engineers.

For most of the students in the slow group we feel we have done well if we have taught them such mastery of the elementary principles of mathematics as they shall need in their life work, and such appreciation of the value of mathematics that the educated public may appreciate its power and the work of its leaders.

We give as much work to the slowest group as to the brightest, but it is simpler. It is drill on fundamentals and thorough ex-

planations of new processes. We have a great deal of review work. In every possible way we overcome as many handicaps to clear thinking as leads to independence in thought. With the slow groups we do not go more slowly than in normal classes; but we stress principles and fundamentals while the other groups take more difficult problems and receive inspiration from discussions of the great power and use of mathematics in the world of industry and science. At present we are trying an experiment of having the slow students in algebra 11B spend more than a semester of time for the course, making the work thorough and adequate for college preparation as so many of the boys in these groups want to be engineers. I have such a class and find that the students master the work well when they have time enough for it. In these groups much more than in other classes we are trying out the individual study plans suggested by Parkhurst and Miller, adapted, of course, to meet the conditions of our school.

The medium group cover practically the same work as the honor group but with much more help from the teacher. Medium classes are made up of students of average ability and a few above normal who because of outside work, ill health, lack of ambition or indifference cannot or will not do superior work. These classes have the honor problems assigned them and they do many of them. At times they have to omit them for the necessary instruction and drill on fundamentals. The encouragement of striving to be an honor group inspires them to their best efforts far more than having among them the superior students who so excel them that they hesitate to test their lesser talents in the presence of the exceptional student.

Among medium classes, as in all groups, we find great differences of ability, yet the students of the group do not differ so widely that they fail to contribute to the good of the group. They really do their best to rank first in it. In the medium classes, if new work is not of unusual difficulty but requires help, we have them work it out together and then have one student develop it when all are held for its development. We use the teach, test, reteach, test idea constantly in all groups. But the medium group requires less reteaching than the the slow group and the honor group least of all.

When the daily test is given and corrected we find it worth

while to go over the work with the group and where students have lost out give individual help. This often gives the stronger students a chance to develop a social sense of helpfulness by aiding the teacher in seeing that all the group have mastery of their work.

The honor group is composed of the students who are bright, alert, eager in mathematics, and make the most of their opportunities in school. They do decidedly original work, depend on themselves in thinking out difficult problems, use the greater initiative they naturally possess. In all groups we develop new processes and principles in algebra and trigonometry, new theorems in geometry in class; but, in the honor groups, the students discover many of the principles and theorems with only a question or suggestion from the teacher. We therefore train them to become independent thinkers much better than we can the medium and slow groups. In these classes we have a chance to discuss related subjects that are highly inspirational to the bright students. We hope to educate some of them so that they will realize that much of art, most of our commerce, all science, is dependent on mathematics, and so that they may be willing to be plodders, patient workers, possibly discoverers of new truth that will be helpful to mankind. The human race can ill afford to let one latent great mathematician fail to discover his talent. Possibly one reason that America has not yet produced a great mathematician, as have the leading countries of Europe, is because she has not given in her schools sufficient attention to the exceptional child. Grouping according to ability helps students to discover themselves in giving those who have ability in mathematics an appreciation of its value and an enthusiasm in its pursuit.

THE FORM OF THE UNIVERSE

By DAVID EUGENE SMITH, LL.D.

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Ancient Ideas.—Closely related to the idea of infinity, in the treatment of which we so neglect our opportunities in the teaching of mathematics in our high schools, is the question of the form of the universe about us. It may not be possible, by casual conversation, to present such a subject to the classes in any satisfactory manner, but we shall teach mathematics with greater earnestness of purpose for having ourselves considered, even though superficially, this and similar problems.

The early philosophers generally looked upon the earth as flat and the heavens as a vast dome. To account for the movement of the planets, the sun, and the moon, they came to imagine a series of concentric crystal domes, one for each of the mobile bodies.

Under the influence of the Greek philosophers they came to recognize that the earth is a sphere, and through the labors of Eratosthenes, in the third century B.C., they came to know to a remarkable degree of accuracy, considering the state of learning at that time, its circumference and diameter. With this recognition of the form of the earth there came the belief that the universe is a sphere, the earth being the center; in other words, there began to be recognized the geocentric theory of a vast spherical region of unfathomable space.

In the Middle Ages this universe was still believed to consist of a series of concentric spheres, the smallest being the abode of air, the planets occupying special spheres of their own, the fixed stars filling another sphere, and finally spheres being imagined for the angels and the Deity. In all early history, however, the universe was either a hemisphere or a sphere, and when the heliocentric or sun-centered theory developed, it merely shifted the center without changing the concept of the general figure of what Petrarch spoke of as "those spacious regions where our fancies roam."

Pascal's Epigram.—Professor Keyser, whose notable monograph *Concerning the Figure and the Dimensions of the Universe of Space* [1] should be read by every teacher of mathematics and by the capable students as well, has spoken of "the familiar and brilliant words of one who in the span of a short life achieved a seven-fold immortality: immortality as a physicist, as a philosopher, as a mathematician, as a theologian, as a writer of prose, as an inventor, and as a fanatic."

This man was Blaise Pascal, one of the greatest geniuses in a country and an age of geniuses, a brilliant product of France and of the seventeenth century.

Among the many happy *mots* of this distinguished writer is the following: "The universe is an infinite sphere whose center is everywhere and whose circumference is nowhere." The words seem, when one first hears them, to have been put together for effect, a mere attempt at creating a striking phrase. But Pascal was not a man to indulge in forms of picturesque deceit; what he said he believed, and he intended others, as their abilities might allow, to understand and believe as well. His works are filled with expressions of awe at the sublimity of the universe about him. "The eternal silence of these infinite spaces," he writes, "fills me with fear." (Le silence éternel de ces espaces infinis m'effraie.) To the thoughts of such a man we may well give heed.

Professor Keyser's Discussion.—Professor Keyser changes the words of his description of the universe to harmonize more perfectly with present usage, and reads the sentence thus: "Space is an infinite sphere whose center is everywhere and whose surface is nowhere," proceeding then to prove the correctness of his statement and to explain his meaning. Chiefly for the purpose of calling attention to his essay and urging its careful study, I shall now endeavor to set forth certain of the leading points in this argument.

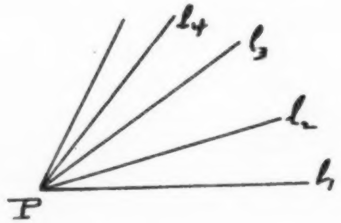
Suppose that from any point P there extend rays $l_1, l_2, l_3, \cdots$, as far as the nature of space allows. The question then arises, do all the rays of this sheaf extend equally far from P? Consider any particular ray, l_1, and imagine milestones placed at the distance of every successive mile from P.

[1] See also his work, *The Human Worth of Rigorous Thinking*, New York, 1925, Chapter V.

Then consider the sequence 1, 2, 3, 4, $\cdots$, extending without end.

"The difference between a sequence that stops somewhere and one that has no end is awful. No one, unless spiritually unborn or dead, can contemplate that gulf without emotions that take hold of the infinite and everlasting." awful?

Suppose that we have set up n mileposts, n being any number *however large.* Then l_1 "admits of precisely as many mileposts as there are cardinal numbers, neither more nor less," and



similarly for $l_2, l_3, \cdots$. Hence $l_1, l_2, l_3, \cdots$ must all be equal, for *we have established a one-to-one correspondence* between the mileposts, which is the essence of equality. Hence *space is a sphere with center P.*

Moreover, this sphere is infinite, for, as stated in a preceding article of the MATHEMATICS TEACHER (Vol. XX, p. 419), the infinite sequence 1, 2, 3, 4, $\cdots$ is such that the whole is equivalent to a part. That is, *space is an infinite sphere with center P.*

Now to consider the question of the center of this sphere, take any point O somewhere in our neighborhood, and about O suppose a sphere to be constructed with any finite radius.

"By saying that the center P is everywhere we mean that P may be taken to be *any* point within a sphere centered at O and having a finite radius. $\cdots$ And by saying that the surface of our infinite sphere is nowhere we mean that no point of the surface can be reached by traveling out from P any *finite* number, however large, of miles."

Hence the center of this sphere is everywhere and its surface is nowhere, these terms being defined in a reasonable and consistent manner.

The Scientific View of the Case.—What Professor Keyser gives us may be called the philosophic view of the case, and we are

naturally led to ask what the scientists have to say with respect to the form of the universe.

When Einstein found it possible to separate out a cosmos of four dimensions in time and space, his formulas showed this space to be spherical, its geodesic lines being circles and the radii of these circles being capable of calculation. The Dutch astronomer de Sitter (1913) calculated the radius of this universe to be approximately 20 million light years, so that the length of a geodesic line would be $2\pi 20$ million light years, or, say, upwards of 125 million light years. A ray of light, therefore, moving along such a line would return to its source again in this period of time, and "just as great circles on a sphere which come from the north pole intersect again at the south pole, so in spherical space each star may have its 'anti sun'; and possibly a part of the stars which we see in the sky may merely be such 'anti suns.' " Such are the wide cosmological horizons which the general theory of relativity opens before us."[2]

In recent years we have come to speak of separate universes in the great cosmos—vast flattened groups of millions of solar systems, each universe at a distance of millions of light years from its nearest neighbor. Such seems to be the general nature of the universe of universes—a sphere of dimensions which transcend our powers of comprehension, but still, as Pascal said, with center everywhere and with surface nowhere. Is this universe of universes itself but an atom in another cosmos? The question is natural; the answer is impossible.

[2] A. V. Vasiliev, *Space, Time, Motion,* translated from the Russian by H. M. Lucas and C. P. Sanger, London, 1924, p. 195. Hereinafter referred to as Vasiliev.

UNDERSTANDING AND PRACTICE

By JOSEPH JABLONOWER

Ethical Culture School, New York City

We want our pupils to have thorough understanding of the subject which we are teaching. But thoroughness is an ideal of perfection and, like any ideal of perfection, it can be approximated only, approximated closely or crudely, according as we are more or less fortunate in our approach and in our treatment of the subject.

Thorough understanding in the sense of exhaustive and complete understanding, of the meaning of a term or a relationship—be it in mathematics or in any other field of knowledge—is impossible. Why?

A concept is the substitute for or the name of an experience or of a class of experiences. The meaningfulness of the concept is therefore a function of the richness of the experience itself, and is subject to continuous change through growth. It is not an exaggeration to say that a concept, like an organism, which is no longer capable of growth is dead—or petrified.

In fact, therefore, thorough understanding, in the sense of exhaustive and complete understanding, is a contradiction in terms. At any given time, it is true, one aspect of a term or of a relationship, that is, one meaning, is emphasized rather than another. That special meaning may be all that is necessary in the given situation. But the teacher should provide always for the expansibility and even the variability of that meaning. Take the concept *number* itself. In the ordinary use of the term we may think of it as the name of a class, the designation of a position in a series, the ratio of one quantity to another, and so on; no one meaning necessarily contradictory to the others, and yet not interchangeable with them in most discussions which involve a logical use of the term. To the mathematician, number has all these meanings, although in a particular discussion he may use the term for the time being in only one of these senses. To the man of the street, for that matter, it has several meanings,

only one of which may be dominant in his mind at any given time.

The meanings of the terms *adding* and *multiplying* are fruitful illustrations of the inexhaustible character of the meanings of terms. For the child in the grades, to add means to increase, so that the sum is quantitatively greater than any addend. When that child gets into his high school work he is inclined to resist, on the basis of his initial "thorough" understanding, the notion that a sum may be less than any of its addends. Similarly, when he first acquires the notion of product, he thinks of it as being greater than any of its factors and he learns that the factors are commutative. But in working with fractions he finds that first understanding of the meaning of product undergoing serious modification. Again, if he should study vector algebra, he learns that the commutative law in the multiplication of certain numbers no longer operates.

Usually the initial, crude concept is sufficient for ordinary purposes, even if it is definitely and demonstrably erroneous. Thus, when we say, as we still do in our less pedantic moments: "The sun moves in an east-west direction," instead of the more correct (so far as we know): "The earth rotates on its axis in the direction known as west to east," there is no misunderstanding or misleading. Within the limits of everyday experience the picture of the sun moving in one direction rather than of the earth moving in the opposite direction produces the same net result and accounts for the phenomena which are the subject of ordinary discourse. The more nearly correct description, so far as present knowledge goes, is necessary only when other and less immediate experiences are involved in the discourse.

The meaning of a term or of a thing is the purpose which it serves in our task of apprehending our environment—physical or social; meaning, therefore, is organic and capable of growth; meaning is inexhaustible.

What significance have these considerations for the teacher and the learner?

1. The name should be the label of an experience, not a substitute for the experience. The Kantian maxim has it: "Concept without percept is empty." The teacher very often accepts the word for the deed, the label for the content. Children are led to acquire early the habit of mistaking glibness for knowledge.

2. We tend (perhaps unavoidably) to generalize particular experiences of the most special kind. The suspended judgment is the characteristic only of the trained mind—and even the trained mind will, as a rule, practice this virtue only in the field in which it acquired the virtue through special training and practice. The scientist will be ludicrously dogmatic in the field of politics or theology and will yet speak with perfect cogency of the tentative character of his judgment in the field where he is really most at home.

In introducing to the child therefore some meaning which we believe is new to him, we might do well to make a survey, through a pre-test, of the knowledge which he already has of the meaning. This pre-test will serve a twofold purpose. First, it will reveal the amount of teaching which it is necessary to do; second, it will reveal misconceptions and errors which must be displaced by the correct or more readily workable meaning. It will often also show that the familiar and the commonplace when considered more carefully take on aspects that make them startlingly unfamiliar or meaningful in a new context. Familiar words and familiar persons have a way of revealing such aspects through a novel context or situation. Such is the case, already referred to, with the fundamental operations of arithmetic as they become elements of the generalized arithmetic known as algebra. Such is the case, too, with postulates, at first seeming only circumlocutions for obvious and trivial facts but later fructifying into a coherent closed system of thought.

3. Knowledge—culture—is therefore in a real sense, in large part, at least, the enrichment of meaning, and the extension of it. In this process of enriching the meaning of a term or of a relation, the original sense of the term or relation plays a very minor rôle, often being, in the new context, only a special application of the general meaning. Take for example the use or meaning of the term exponent. In the elementary grades 3^2 means the very concrete experience of taking 3 as a factor twice. But to the pupil who understands the exponent only in this sense, $3^{1/2}$ or 3^{-2} is abstract, completely devoid of meaning, absurd. To the pupil whose understanding of exponent has been enriched beyond the first elementary stage, later forms are concrete, meaningful, reasonable. Meanings are not in themselves abstract or concrete. As Professor Dewey reminds us in his book "How

We Think," meanings which are abstract may, through use, become concrete, and, per contra, meanings which are at first concrete may, upon closer examination, reveal elements that are strange and unfamiliar—and to that extent abstract.

4. This functional view of meaning has an important corollary: Conscious practice in the use of a meaning increases its concreteness. Merely defining a meaning in terms of synonyms may leave the meaning as abstract as ever. Things have meaning for us in proportion as they are tools in our doing or thinking. "Learn to know by using" is a safe maxim in the field of thinking no less than in the field of conduct. The best definition of a spiral staircase is still the irrepressible gesture of persons whose nervous and muscular coordinations are up to par. The person who is unable to walk or to use his arms cannot have the adequate understanding of the term.

5. But the doing must necessarily be suited to the stage of the development of the learner. The degree of thoroughness, if the term is understood in the sense indicated above, must be related to the psychological development of the learner. The degree of thoroughness, it cannot be emphasized too strongly, is dependent also upon the extent of the practice. The child who has had more practice with the simpler uses of exponents or with the simpler applications of, say, the binomial theorem, understands the meaning of these better than does the child who has had less practice, with the proviso, of course, that in this as in many other fields there may be a law of diminishing returns at work, and that practice without understanding is not the kind of practice we mean, for to quote the other half of the Kantian maxim: "Percept without concept is blind." Practice makes the understanding vivid.

6. As far as possible the meaning of a term or a relation at any given stage in the life of the learner should not be such as to render it incapable of growth or expansion as the pupil acquires experiences which call for such growth or expansion. To recur to our example of the spiral staircase, a later and more mature need of the analysis of the term, spiral, may lead to a study of the helix, to an analytic study of its properties, to a relation of it to phenomena of electricity and magnetism and of other aspects of it in vector analysis.

To summarize: in all teaching it is a question of how long a

start we must take before we deal with the particular situation which is the initial concern. We are always faced with two possible extremes. The one is that we may take too big a start in the preparation of background, very much as does Washington Irving's historian in *Knickerbocker's History of New York.* The other is that we may plunge the child so unprepared into the use of terms that nothing but malappropriateness results. This is illustrated in the experience of a little girl who could not understand why the hymn which she was learning should refer to a consecrated "cross-eyed" bear.

The late Professor G. Stanley Hall and other students in the field of psychology have called attention to the need we have of taking stock of the concepts of our pupils. A careful study of the manner in which habits are formed will impress us with the importance of analyzing for ourselves the novel aspects which given concepts may have for the learner at any stage, and the manner in which these novel aspects are most healthfully assimilated. A proper appreciation of the functional aspect of all meaning will prompt us as teachers to suit the aspects of the meaning to the psychological readiness of the child to assimilate the meaning. A reasonable care in economy of learning will prompt us to introduce, at any stage of the learner's development, as far as possible, only those aspects of meaning of terms which are consonant with later and more comprehensive aspects of meaning of those terms. (Hence the need for scholarship in teachers.) At all stages, practice through use increases and makes vivid the understanding of meaning.

NEWS NOTES

Report of the Mathematics Section

Central Association of Science and Mathematics Teachers

Detroit Meeting, November 25, 1927, Cass Technical High School

The meeting was called to order by the chairman, Joseph A. Nyberg, Hyde Park High School, Chicago. In the absence of the secretary, Margaret Dady, Waukegan Township High School, Waukegan, Illinois, the chairman appointed Edwin W. Schreiber of Ann Arbor as secretary pro tempore, who read the minutes of the 1926 meeting held at Crane Junior College, Chicago. Mr. Nyberg then introduced Miss Hildegarde Beck, Northwestern High School, Detroit, who presented a paper entitled: "Adjusting the Course of Study in Ninth Grade Mathematics to the Ability of the Pupil." Mr. Charles A. Stone, University High School, Chicago, followed with a paper on: "Functional Analysis of a Unit of Work in Ninth Grade Mathematics." The third paper was presented by Mr. Fred A. Burroughs, John Adams High School, Cleveland, Ohio, entitled: "An Investigation of Achievements in Plane Geometry."

The chairman appointed a nominating committee of three: Prof. Theo. Lindquist, Ypsilanti; Prof. John R. Clark, New York University; and Mr. J. V. McNally, Detroit. The committee submitted the following report which was unanimously accepted: Edwin W. Schreiber, Chairman, Graduate Student, University of Michigan; Martha Hildebrandt, Vice-Chairman, Proviso Township High School, Maywood, Ill.; Margaret Dady, Secretary, Waukegan Township High School, Waukegan, Illinois.

More than sixty persons were in attendance at the Mathematics Section. Thirteen members of the Central Association were present and forty-five visitors registered.

The program of the December meeting of the Buffalo Section of the Association of Teachers of Mathematics in Maryland included: (1) "The Sextant," by Ray W. Spear, Bennett High School; (2) "The Seismograph," by Father Joseph McAree, S.J., Canisius College; and (3) "The Vernier," by Harold S. Fisher, East High School. (Miss Mary Kenny is Secretary of the Section.)

The program of the December meeting of the Association of the Teachers of Mathematics in New England included: (1) "Geometry Notes," by Murtach M. S. Moriarity, Holyoke High School; (2) "The Tetrahedron," by Professor Julian L. Coolidge, Harvard University; (3) "Algebra in the Making," by Elmer R. Bowker, Public Latin School, Boston; and (4) "The Why and the How in Algebra," by Harry B. Marsh, Principal, Technical High School, Springfield, Mass.

Professor Ralph Beatley, of the Harvard Graduate School of Education, will visit three mid-western colleges during the second half of this academic year as Exchange Professor. He will spend February and part of March at Knox College, Galesburg, Illinois; a month at Beloit in Wisconsin; and a month at Colorado College, Colorado Springs. He will lecture on topics of broad educational significance, referring for illustrative material in the main to elementary mathematics.

Members of the Council will be interested in the following tabular representation of the mailing list by states for the November and January issues of the MATHEMATICS TEACHER. In every state except five the number of subscriptions has increased in the two-month period. The increase is notable in several states as can be seen from the table. The total number of subscribers has increased from 3437 in November to 4344 in January—an increase of almost 1000.

	Nov.	*Jan.*		*Nov.*	*Jan.*
Alabama	65	76	North Carolina	70	93
Arizona	9	10	North Dakota	10	12
Arkansas	23	26	Nevada	2	2
California	145	157	New Hampshire	19	21
Colorado	46	57	New Jersey	110	124
Connecticut	69	78	New Mexico	7	9
Delaware	3	3	New York	356	568
Florida	24	32	Ohio	165	223
Georgia	39	38	Oklahoma	49	56
Idaho	12	14	Oregon	27	24
Illinois	302	340	Pennsylvania	279	374
Indiana	125	143	Rhode Island	17	24
Iowa	89	99	South Carolina	21	23
Kansas	136	178	South Dakota	17	20
Kentucky	30	34	Tennessee	20	29
Louisiana	39	42	Texas	136	161
Maine	25	30	Utah	5	5
Maryland	79	89	Vermont	14	17
Massachusetts	235	263	Virginia	38	49
Michigan	108	150	Washington	25	33
Minnesota	70	112	West Virginia	28	36
Mississippi	27	39	Wisconsin	102	125
Missouri	58	88	Wyoming	11	14
Montana	9	11	Philippine Islands	12	14
Nebraska	52	74	Foreign Countries	96	103

It is the hope of the editors that members in those states where the membership is at a standstill, or has decreased, will start a movement to increase their membership materially. The editors will be glad to send advertising material to any who request it.

PROGRAM OF THE NINTH ANNUAL MEETING OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, AT HOTEL STATLER, PARK SQUARE AT ARLINGTON STREET, BOSTON, MASSACHUSETTS, FEBRUARY 24 AND 25, 1928

GENERAL THEME: MATHEMATICS AND MODERN LIFE

Friday Morning Session, 9.45 A.M.

At Hotel Statler, Parlor E

Joint meeting of the Executive and Local Committees. (Important items of business will be discussed at this meeting. Every member of each committee is urged to attend.)

Friday Afternoon Session, 2.00 P.M.

At Hotel Statler, Parlor C

Business Session.
4:00 P.M. Reception and tea, Massachusetts Institute of Technology.

Friday Evening Session, 7.45 P.M.

At Hotel Statler, Georgian Room

Addresses of Welcome:
Wm. B. Snow, Boston, Mass., Assistant Superintendent of Schools.
W. L. Vosburgh, Teachers College, Boston, Mass., President New England Association of Teachers of Mathematics.
Response:
Orpha E. Worden, Teachers College, Detroit, Michigan.
Keynote Address—Mathematics and Modern Life: George D. Olds, formerly President of Amherst College.

Saturday Morning Session, 9.00 A.M.

At Hotel Statler, Georgian Room

Theme: Mathematics in Science and Modern Inventions
Mathematics in Modern Science: H. W. Tyler, Massachusetts Institute of Technology.

Mathematics in Modern Business: Edith Clarke, Schenectady, New York, Engineer General Electric Co.
Discussion.
Presentation of the Third Yearbook: W. D. Reeve, Teachers College, Columbia University, New York.
Discussion.

SATURDAY AFTERNOON SESSION, 2.00 P.M.

At Hotel Statler, Georgian Room

Theme: The Challenge of Modern Life to Teachers of Mathematics

The New Mathematics as a Part of the New Education.

1. Its Nature and Function: Walter F. Downey, Headmaster English High School, Boston, Mass.
2. Its Methods and Devices: Olive A. Kee, Teachers College, Boston, Mass.
3. Its Challenge and Opportunity: Harry C. Barber, Exeter, New Hampshire.

Practical Applications of High School Mathematics: W. S. Schlauch, High School of Commerce, New York.
Discussion.

SATURDAY EVENING SESSION, 6.00 P.M.

At Hotel Statler, Parlors A and B

Annual banquet.

Members and their friends are requested to send their reservations for the dinner to Mr. Harry C. Barber, 76 Court Street, Exeter, New Hampshire. The price of the banquet is three dollars and fifty cents a plate.

Address—Mathematics and Sunshine: H. E. Slaught, University of Chicago.

ANNOUNCEMENT

To Members of the National Council:

National Council of Teachers of Mathematics Third Yearbook on Selected Topics in Teaching Mathematics.

The Third Yearbook will be ready February 25, 1928, and can be secured by sending $1.75 for a bound volume to the Bureau of Publications, Teachers College, 525 West 120th Street, New York City. Send in your order at once.

Table of Contents: Chapter (1) The Fallacy of Treating School Subjects as "Tool Subjects," Charles H. Judd; (2) Mathematics in the Training for Citizenship, David Eugene Smith; (3) Mathematics as an Interpreter of Life, W. S. Schlauch; (4) The Reality of Mathematical Processes, E. R. Hedrick; (5) Developing Functional Thinking in Secondary School Mathematics, E. R. Breslich; (6) Dynamic Symmetry, Marie Gugle; (7) Introductory Calculus as a High School Subject, M. A. Nordgaard; (8) Selected Topics in Calculus for the High School, John A. Swenson; (9) Teaching Thrift through the School Savings Bank, Clifford H. Upton; (10) The Teaching of Direct Measurement in the Junior High School, William Betz; (11) Measurement and Computation, George W. Finley; (12) Problem Solving in Arithmetic, Lucie I. Dower; (13) The Use of Measuring Instruments in Teaching Mathematics, C. N. Shuster; (14) A Mathematical Atmosphere, Olive A. Kee.